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Lagrangian presymplectic constraint analysis of mechanical models of field theories coupled to external fields

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Abstract. The Lecanda-Román-Roy geometric constraint algorithm for presymplectic Lagrangian systems is applied to a mechanical model of singular field theories coupled to time independent external fields. The simple, yet intrinsic structure of the algorithm allows the influence of the external field to be traced through the constraint analysis, showing clearly where pathologies arise—namely in the second generation non-dynamical constraints arising from the stability of the first generation compatibility constraints. Using a coordinate independent geometric algorithm provides a more systematic tool for investigating singular field theories than the usual *ad hoc* manipulation of the field equations; where the essential structure is often obscured by the details of the representation of the model and the complexity of the algebra.

1. Introduction

In recent years there has been steady progress towards the geometric formulation of the theory of dynamical systems with constraints. Since the seminal work of Gotay *et al* [1] on the geometrization of the classic Dirac-Bergmann constraint algorithm for singular Hamiltonian systems and its subsequent partial extension to Lagrangian systems [2, 3], successive authors have refined the process of geometrizing both the constraint algorithm and the connection between the Lagrangian and Hamiltonian formalism for constrained systems with a finite number of degrees of freedom. An excellent modern review of the current situation has been given by Cariñena [4].

The extension of this geometrization programme to infinite dimensional systems such as field theories is less advanced, particularly in the sense of practical utility. Gotay and Nester considered the cases of the free Maxwell and Proca fields [1, 5] using functional derivatives, and much work has appeared on the theoretical aspects of geometrization of infinite dimensional systems [6-9]. One such development has been the use of the multisymplectic formalism as a basis for the geometrization of field theories [10, 11] and of course, the fibre bundle structure of Yang-Mills gauge field theories is well understood. However, these approaches are still some way off a geometrical understanding of the structure and properties of classical field theories and their various pathologies such as acausality and indefiniteness of quantization of coupled higher spin theories [12, 13]. The nature and relation of these pathologies to the structure of the field theory has not been satisfactorily clarified. Although it is known, for example, that the Velo-Zwanzinger and Johnson-Sudarshan problems have

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a common origin in the constraint analysis [14, 16], a thorough understanding of the situation has been hampered by the lack of a systematic, intrinsic, coordinate independent constraint algorithm through which to trace the advent of the pathologies. Few general patterns emerge from the complicated calculations which are necessary in such theories.

In [14, 15] Capri and Kobayashi avoid much of the drudgery of high spin analysis by using a mechanical analogue of a typical higher spin theory, allowing a constraint treatment as for a system with a finite number of degrees of freedom. This allows investigation of the general structure of a high spin theory and its couplings, while dispensing with the trappings and complexities of field theory. This analogue lends itself readily to a geometrical formulation. In [14] the approach is by the Dirac-Bergmann Hamiltonian constraint analysis, whereas [15] uses the more usual direct Lagrangian approach of field theory. While the Dirac-Bergmann Hamiltonian constraint analysis was geometrized some time ago [1], a geometric formulation of a presymplectic, completely Lagrangian constraint analysis has only recently been perfected by Lecanda and Román-Roy [17]. In this paper we apply the Lecanda-Román-Roy geometric algorithm to the mechanical model of Capri-Kobayashi, identifying the stage at which pathologies arise. The advantage of this approach is that it sets such difficulties in a general geometric framework of a systematic constraint analysis. In terms of the Lecanda-Román-Roy algorithm the invertibility condition of Capri-Kobayashi arises in the stability of the compatibility conditions generating the first generation dynamical constraints. This stability generates a second generation of dynamical constraints, the stability of which determines the vector field on the final constraint submanifold uniquely, only if the invertibility condition is satisfied. The causality of the flow of this vector field is then dependent on the invertibility conditions and furthermore, this condition dictates the definiteness of the quantization procedure as performed on the constraint submanifold. Thus, in general, such pathologies as acausality and quantization indefiniteness are pre-empted by external field induced disruptions in the constraint analysis [16].

In section 2 we describe the Lecanda-Román-Roy formalism and this is applied to the Capri-Kobayashi model in section 3. The results are discussed in section 4.

2. A presymplectic constraint algorithm for Lagrangian systems

The Lecanda-Román-Roy (LR) algorithm analyses singular Lagrangian systems with a finite number of degrees of freedom, to determine a submanifold of the velocity phase space, with a tangent vector field thereon which is a solution of the Lagrangian equations of motion and also a second order differential equation. All stages of the algorithm reside in the Lagrangian domain, with no excursion into the Hamiltonian formalism.

Consider a system with a configuration space Q, velocity phase space TQ, and with an almost-regular [2] Lagrangian L. We introduce the vertical endomorphism J (called the almost tangent structure by Gotay-Nester [2, 3] and often denoted by S, as for example in [4, 18]) as a pointwise map:

$$J: T(TQ) \to V(TQ)$$

where $V(TQ)$ is the set of vertical vectors on TQ, given in local coordinates, $(q, v) \in TQ$

by

$$J = \frac{\partial}{\partial v^i} \otimes dq^i \tag{2.1}$$

(for intrinsic definition see [4]).

Defining the Lagrangian 1-form $\theta = dL \circ J$ on TQ we can then construct the usual Lagrangian presymplectic 2-form

$$\omega = -d\theta.$$

In local coordinates (q, v):

$$\theta = \sum_i \frac{\partial L}{\partial v^i} dq^i \tag{2.2}$$

$$\omega = \sum_{i,j} \frac{\partial^2 L}{\partial q^j \partial v^i} dq^i \wedge dq^j + \sum_{i,j} \frac{\partial^2 L}{\partial v^j \partial v^i} dq^i \wedge dv^j. \tag{2.3}$$

In terms of local coordinate system (q_i, v_i), ω may be regarded as a transformation defined by contraction $\hat{\omega}$: $\mathscr{F}(TQ) \to \Lambda^1(TQ)$, represented by a matrix [4]

$$\hat{\omega} = \begin{bmatrix} A & -W \\ W & 0 \end{bmatrix} \tag{2.4}$$

where

$$A_{ij} = \frac{\partial^2 L}{\partial q^i \partial v^j} - \frac{\partial^2 L}{\partial v^i \partial q^i}$$

$$W_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j}.$$

In the usual manner we define the energy, E, by

$$E = \Delta(L) - L$$

where Δ is the Liouville vector field in TQ which generates dilations along the fibres:

$$(\Delta f)(q, v) = \frac{d}{dt} f(q, e^t v)|_{t=0}$$

for $f \in C^\infty(TQ)$. In local coordinates (q, v):

$$\Delta = v^i \frac{\partial}{\partial v_i}.$$

For a physical system we require the Lagrange equations resulting from the action principle for E to yield second-order differential equations (SODE) of motion. Geometrically, a SODE is a vector field X whose integral curves are canonical lifts to TQ of curves on Q. In local coordinates the general form of such a vector field is

$$X = v^i \frac{\partial}{\partial q^i} + f^i(q, v) \frac{\partial}{\partial v_i}$$

yielding integral curves

$$\frac{dq^i}{dt} = v^i$$

$$\frac{dv^i}{dt} = f^i(q, v).$$

Using the coordinate form for J and Δ the geometrical condition for X to be SODE is seen to be $JX = \Delta$.

The LR algorithm analyses the results of the action principle for singular Lagrangians by searching for a vector field $X \in \mathscr{F}(TQ)$ (the set of vector fields on TQ) and a submanifold $S \subset TQ$ such that

(i) The Lagrange equations

$$i_X \omega = \mathrm{d}E \tag{2.5}$$

are satisfied on restriction to S, where E is the energy.

(ii) X is a second-order differential equation (SODE) on S.

(iii) X is tangent to S.

We will take for granted all the requirements of good behaviour of those submanifolds of TQ with which we have to deal, except in those circumstances in which pathologies due to external field values may disrupt this behaviour. For example, a basic assumption of the LR algorithm is that dim(ker ω) is constant, but in general this will depend on external field values.

The first stage in the algorithm is to derive from (2.5) the first generation constraints, which are of two types—dynamical, coming from the compatibility conditions of (2.5) and non-dynamical, arising from the SODE conditions on the solutions of (2.5). This yields a submanifold $S_1 \subset TQ$ and a family of vector fields:

$$\{X_0 + Y_0 + V : V \in V(\ker \omega)\} \tag{2.6}$$

such that

$$(i_{(X_0+Y_0+V)}\omega - \mathrm{d}E)|_{S_1} = 0 \tag{2.7}$$

and

$$X_0 + Y_0 + V \text{ are SODE} \tag{2.8}$$

where $V(\ker \omega)$ denotes the vertical part of $\ker \omega$:

$$V(\ker \omega) = \ker \omega \cap V(TQ).$$

In (2.6), X_0 is a particular solution of the Lagrangian equations, while Y_0 is an element of $\ker \omega$ chosen to ensure that $X_0 + Y_0$ is SODE on S_1. V on the other hand is an arbitrary element of $V(\ker \omega)$.

The submanifold S_1 is defined by

$$S_1 = \{x \in TQ; (i_Z(i_D\omega - \mathrm{d}E))(x) = 0 \;\forall Z \in \mathscr{M}\} \tag{2.9}$$

for any SODE D, where

$$\mathscr{M} = \{Z \in \mathscr{F}(TQ); JZ \in V(\ker \omega)\} \tag{2.10}$$

is essentially the inverse image of $V(\ker \omega)$ under the vertical endomorphism J.

In terms of the matrix representation (2.4) of ω the general form of $Z \in \mathscr{M}$ is [4]

$$Z = \xi_i \frac{\partial}{\partial q^i} + \gamma_i \frac{\partial}{\partial v^i} \tag{2.11}$$

where

$$W\xi = 0$$

$$\gamma = \text{arbitrary}.$$

The two types of first generation constraints referred to above, which specify the submanifold S_1, are defined by:

Dynamical constraints

$$\zeta^1 \equiv i_Z \, dE = 0 \qquad Z \in \ker \omega. \tag{2.12}$$

Non-dynamical constraints

$$\eta^1 \equiv i_Z i_Y \omega = 0 \tag{2.13}$$

where

$$Z \in \mathcal{M} \qquad Z \notin \ker \omega \qquad Y \in \mathcal{F}(TQ) \text{ with } X_0 + Y \text{ a SODE.}$$

In practical terms the two types of constraint differ in their $\hat{F}L$-projectibility. The dynamical constraints are $\hat{F}L$-projectable to corresponding constraints in the Hamiltonian formalism. Their stability (see below) may lead to further generations of both dynamical and non-dynamical constraints. The non-dynamical constraints are not $\hat{F}L$-projectable and their stability can lead only to elimination of degrees of freedom in the vector field solution, rather than to further constraints.

In general the family of vector fields (2.6) is not tangent to the submanifold S_1 and will therefore evolve flows off this submanifold (i.e. the constraints are not perserved in time). The requirement of such tangency yields further *stability conditions*, which define a new submanifold $S_2 \subset S_1$. These arise from ensuring that the first generation constraints ζ^1, η^1 are stable under the vector fields in (2.6). As a consequence of their $\hat{F}L$-projectibility this yields possible new second generation constraints from the dynamical first generation constraints, ζ^1:

$$((X_0 + Y_0)(i_Z \, dE))(x) = 0 \qquad \forall Z \in \ker \omega \tag{2.14}$$

defining a new submanifold $S_2 \subset S_1$. The non-dynamical first generation constraints η^1 on the other hand yield non-trivial equations

$$\begin{gathered} ((X_0 + Y_0)\omega(Z, Y))(x) + (V\omega(Z, Y))(x) = 0 \\ \forall Z \in \mathcal{M} \qquad Z \notin \ker \omega \qquad Y \in \mathcal{F}(TQ) \qquad X_0 + Y \text{ SODE} \end{gathered} \tag{2.15}$$

which may determine some or all of the otherwise arbitrary $V \in V(\ker \omega)$, rather than producing further constriants. The remaining undetermined V satisfy

$$V\omega(Z, Y) = 0 \qquad \forall Z \in \mathcal{M}. \tag{2.16}$$

So we may now write the vector field solution on the new submanifold S_2 defined by (2.14) in the form $X_0 + Y_0 + V_0 + V'$ where $X_0 + Y_0 + V_0$ is tangent to S_1 on all points of S_2, V_0 is a particular solution of (2.15) and $V' \in V(\ker \omega)$ is an arbitrary solution of (2.16) which is tangent to S_1 on all points of S_2. S_2 is the submanifold of second generation constraints, defined by the constraint functions

$$(X_0 + Y_0)(i_Z \, dE) \qquad \text{for } Z \in \ker \omega. \tag{2.17}$$

Now, of course, we must ensure that the new solutions $X_0 + Y_0 + V_0 + V'$ are tangent to S_2. For this purpose the second generation constraints must be again split into dynamical (i.e. $\hat{F}L$-projectable) and non-dynamical (non $\hat{F}L$-projectable). It is shown in [17] that the second generation dynamical constraints are those defined by

$$\zeta^2 \equiv X_0(i_Z \, dE) \qquad Z \in \ker \omega \tag{2.18}$$

such that

$$Y(i_Z\,\mathrm{d}E)|_{S_1}=0 \qquad \forall Y\in\ker\omega$$

while the non-dynamical constraints are given by

$$\eta^2=(X_0+Y_0)i_Z\,\mathrm{d}E \qquad Z\in\ker\omega \tag{2.19}$$

such that

$$Y_0 i_Z\,\mathrm{d}E|_{S_1}\neq 0.$$

Having thus split the second generation constraints we can now investigate stability exactly as for the first generation constraints. Demanding stability of the ζ^2 under $X_0+Y_0+V_0+V'$ yields possibly new third generation constraints, defining a new submanifold S_3, while stability of the η^2 can only yield further determination of the remaining arbitrary V'.

The third generation dynamical constraints arising from the stability conditions for the dynamical second generation constraints ζ^2 may be written

$$\zeta^3\equiv(X_0+Y_0)(X_0(i_Z\,\mathrm{d}E))=0 \qquad Z\in\ker\omega$$

where

$$Y_0 i_Z\,\mathrm{d}E|_{S_1}=0$$

and these define $S_3\subset S_2$:

$$S_3=\{x\in S_2;\,((X_0+Y_0)(X_0(i_Z\,\mathrm{d}E)))(x)=0\;\forall Z\in\ker\omega,\;Y_0 i_Z\,\mathrm{d}E|_{S_2}=0\}$$

which is the third generation constraint submanifold.

The stability condition for the non-dynamical second generation constraints $\eta^2=(X_0+Y_0)i_Z\,\mathrm{d}E$, $Z\in\ker\omega$, $Y_0 i_Z\,\mathrm{d}E|_{S_1}\neq 0$ yields new equations for the V':

$$V'(X_0+Y_0)i_Z\,\mathrm{d}E=-(X_0+Y_0+V_0)(X_0+Y_0)i_Z\,\mathrm{d}E$$

thereby reducing further any gauge degrees of freedom.

The above process may be iterated as necessary, concluding in (for a physically sensible system) a non-trivial final constraint submanifold S_k on which there exist non-trivial SODE solutions, D, tangent to S_k, which have the general form $X_0+Y_0+V_\delta+V$ where $V\in V(\ker\omega)$ and V_δ is completely determined while, V, also tangent to S_k, represents the remaining gauge freedom. At this point the presymplectic Lagrangian constraint algorithm terminates. See [17] for further details and a general characterization of the ith iteration in terms of vector fields $T\in\mathscr{F}(S_i)^\perp$, which we will not be using here.

3. The mechanical constraint model

We consider a Lagrangian of the form

$$L=\dot\phi_a^* m_{ab}\dot\phi_b+\dot\phi_a^* c_{ab}\phi_b-\phi_a^*\bar c_{ab}\dot\phi_b-\phi_a^* r_{ab}\phi_b \tag{3.1}$$

where the ϕ_a, ϕ_b^*, a, $b=1,2,\ldots,N$ are regarded as generalized coordinates in a $2N$-dimensional configuration space Q. We will adopt the convenient physicists practice of regarding ϕ_a, ϕ_b^* as independent coordinates, which also provides a useful check on the final outcome of calculations. Many bosonic relativistic field theories, including gauge and high spin theories, may be cast in the form of (3.1) by a '3 + 1 decomposition' wherein spatial derivatives and external fields are tucked away in the c_{ab} and r_{ab}. Here however, we will follow Capri and Kobayashi [14, 15] and regard (3.1) simply as a mechanical analogue of such theories, with similar constraint structure. In order to apply the LR formalism the coefficients m_{ab}, c_{ab}, $\bar{c}_{ab}$, r_{ab} are assumed to be time independent. Reality of the Lagrangian demands that m_{ab}, r_{ab} are Hermitian, while the c_{ab} satisfy

$$\bar{c}^*_{ab} = -c_{ba}.$$

It is possible to choose $\bar{c}_{ab} = c_{ab}$ of course, then c_{ab} would be anti-Hermitian, but in practice (as for example in the Proca theory coupled to an electromagnetic field, [15]) the most usual covariant form of the field theory Lagrangian may not reduce directly to this form, so it is convenient to retain the generality of (3.1).

Being Hermitian m_{ab} is diagonalizable and we will therefore take it to be so and work from a Lagrangian of the form

$$L = m_a \dot{\phi}^*_a \delta_{ab} \dot{\phi}_b + \dot{\phi}^*_a c_{ab} \phi_b - \phi^*_a \bar{c}_{ab} \dot{\phi}_b - \phi^*_a r_{ab} \phi_b \tag{3.2}$$

to which we now apply the formalism of section 2.

If all $m_a \neq 0$ then (3.2) yields no constraints, so we assume that

$$\begin{aligned} &m_a = 0 \qquad a = i, j, k, \ldots = 1, 2, \ldots, (N-r) \\ &m_a \neq 0 \qquad a = \alpha, \beta, \gamma, \ldots = (N-r+1), \ldots, N \end{aligned} \tag{3.3}$$

and rewrite the Lagrangian consistent with this split as

$$\begin{aligned} L = \sum_\alpha m_\alpha \dot{\phi}^*_\alpha \delta_{\alpha\beta} \dot{\phi}_\beta + \dot{\phi}^*_i c_{ij} \phi_j + \dot{\phi}^*_i c_{i\alpha} \phi_\alpha + \dot{\phi}^*_\alpha c_{\alpha i} \phi_i + \dot{\phi}^*_\alpha c_{\alpha\beta} \phi_\beta \\ - \phi^*_i \bar{c}_{ij} \dot{\phi}_j - \phi^*_i \bar{c}_{i\alpha} \dot{\phi}_\alpha - \phi^*_\alpha \bar{c}_{\alpha i} \dot{\phi}_i - \phi^*_\alpha \bar{c}_{\alpha\beta} \dot{\phi}_\beta + \phi^*_a r_{ab} \phi_b. \end{aligned} \tag{3.4}$$

The Lagrangian presymplectic 2-form ω is found to be, from (2.3):

$$\begin{aligned} \omega = m_\alpha \delta_{\alpha\beta}\, \mathrm{d}\phi^*_\alpha \wedge \mathrm{d}\dot{\phi}_\beta + m_\alpha \delta_{\alpha\beta}\, \mathrm{d}\phi_\alpha \wedge \mathrm{d}\dot{\phi}^*_\beta + d_{ji}\, \mathrm{d}\phi^*_j \wedge \mathrm{d}\phi_i + d_{\alpha i}\, \mathrm{d}\phi^*_\alpha \wedge \mathrm{d}\phi_i \\ + (-d_{i\alpha})\, \mathrm{d}\phi_\alpha \wedge \mathrm{d}\phi^*_i + d_{\beta\alpha}\, \mathrm{d}\phi^*_\beta \wedge \mathrm{d}\phi_\alpha \end{aligned} \tag{3.5}$$

where

$$d_{ab} = c_{ab} + \bar{c}_{ab}.$$

Let

$$K = k_i \frac{\partial}{\partial \phi_i} + k_\alpha \frac{\partial}{\partial \phi_\alpha} + K_i \frac{\partial}{\partial \dot{\phi}_i} + K_\alpha \frac{\partial}{\partial \dot{\phi}_\alpha} + \mathrm{cc}$$

(CC denotes corresponding conjugate terms such as $k^*_i \partial/\partial\phi^*_i$) be a typical element of $\ker \omega$. Then K satisfies

$$\begin{aligned} &d_{ji} k^*_j + d_{\alpha i} k^*_\alpha = 0 \\ &-d_{ij} k_j + d_{i\alpha} k_\alpha = 0 \\ &-m_\alpha K^*_\alpha + d_{i\alpha} k^*_i + d_{\beta\alpha} k^*_\beta = 0 \\ &-m_\alpha K_\alpha - d_{\alpha i} k_i - d_{\alpha\beta} k_\beta = 0 \\ &m_\alpha k^*_\alpha = 0 \\ &m_\alpha k_\alpha = 0. \end{aligned} \tag{3.6}$$

(It is hoped the occasional abeyance of the summation convention will be clear from the context.)

The solution of (3.6) is

$$k_\alpha = k_\alpha^* = 0$$
$$d_{ji}k_j^* = 0$$
$$d_{ij}k_j = 0$$
$$m_\alpha K_\alpha^* = d_{i\alpha}k_i^*$$
$$m_\alpha K_\alpha = -d_{\alpha i}k_i$$
$$K_i, K_i^* \qquad \text{arbitrary.}$$

The nature of ker ω is clearly dependent on the sub-matrix d_{ij}. Specifically

$$\dim(\ker\omega) = 4(N-r) - 2\,\mathrm{rank}(d_{ij}).$$

Since d_{ij} may contain parameters in the form of external fields it may be that dim(ker ω) varies with the external field. This would imply a constraint structure dependent on the external field and so we will exclude this possibility by insisting that rank(d_{ij}) is constant, independent of the value of the external field.

There are then two possibilities with distinct constraint structures rank$(d_{ij}) = N-r$ and rank$(d_{ij}) = r_1 < N-r$.

rank$(d_{ij}) = N-r$

In this case d_{ij} is non-singular and the solution to (3.6) yields

$$k_i = k_i^* = 0$$
$$k_\alpha = k_\alpha^* = 0$$
$$K_\alpha = K_\alpha^* = 0$$
$$K_i, K_i^* \qquad \text{arbitrary.}$$

Thus in the case $|d_{ij}| \neq 0$ a basis for ker ω is

$$\left\{ W_i = \frac{\partial}{\partial \dot\phi_i},\ W_i^* = \frac{\partial}{\partial \dot\phi_i^*} \right\}. \tag{3.7}$$

From (2.11) we find that a local basis for $\mathcal{M}$ is

$$\mathcal{M} = \left\{ Z_i = \frac{\partial}{\partial \phi_i}, Z_i^* = \frac{\partial}{\partial \phi_i^*};\ W_a = \frac{\partial}{\partial \dot\phi_a},\ W_a^* = \frac{\partial}{\partial \dot\phi_a^*} \right\}. \tag{3.8}$$

There are no compatibility constraints, as $i_Z\,\mathrm{d}E$ is identically zero for $Z \in \ker\omega$. Now let

$$X = a_i \frac{\partial}{\partial \phi_i} + a_\alpha \frac{\partial}{\partial \phi_\alpha} + A_i \frac{\partial}{\partial \dot\phi_i} + A_\alpha \frac{\partial}{\partial \dot\phi_\alpha} + \mathrm{cc} \tag{3.9}$$

be a typical vector field solution of the Lagrange equations (2.5). Then the components of X satisfy the equations

$$\begin{aligned} a_\alpha^* &= \dot\phi_\alpha^* \\ a_\alpha &= \dot\phi_\alpha \end{aligned} \tag{3.10}$$

$$\begin{aligned} d_{ji}a_j^* + d_{\alpha i}a_\alpha^* &= r_{ji}\phi_j^* + r_{\alpha i}\phi_\alpha^* \\ -d_{ji}a_j - d_{i\alpha}a_\alpha &= r_{ij}\phi_j + r_{i\alpha}\phi_\alpha \end{aligned} \tag{3.11}$$

$$\begin{aligned} -m_\alpha A_\alpha^* + d_{i\alpha}a_i^* + d_{\beta\alpha}a_\beta^* &= r_{i\alpha}\phi_i^* + r_{\beta\alpha}\phi_\beta^* \\ -m_\alpha A_\alpha - d_{\alpha i}a_i - d_{\alpha\beta}a_\beta &= r_{\alpha i}\phi_i + r_{\alpha\beta}\phi_\beta. \end{aligned} \tag{3.12}$$

These equations determine a_α, a^*_α and thence a_i, a^*_i, since $|d_{ij}|\neq 0$, in terms of the velocities $\dot\phi_\alpha$, $\dot\phi^*_\alpha$ and the coordinates ϕ_a, ϕ^*_a. Then (3.12) determines A_α, A^*_α but A_i, A^*_i remain undetermined.

Now applying the SODE condition (2.9), using $\mathcal{M}$ given by (3.8) yields the first generation non-dynamical constraints

$$\begin{aligned}\eta^{*1}_i &= d_{ji}\dot\phi^*_j + d_{\alpha i}\dot\phi^*_\alpha - r_{ai}\phi^*_a = 0\\ \eta^1_i &= d_{ij}\dot\phi_j + d_{i\alpha}\dot\phi_\alpha + r_{ia}\phi_a = 0\end{aligned}\tag{3.13}$$

which, by comparison with (3.11) guarantee that

$$a_i = \dot\phi_i \qquad a^*_i = \dot\phi^*_i$$

and hence that (3.9) is SODE:

$$X = D = \dot\phi_i\frac{\partial}{\partial\phi_i} + \dot\phi_\alpha\frac{\partial}{\partial\phi_\alpha} + A_i\frac{\partial}{\partial\dot\phi_i} + A_\alpha\frac{\partial}{\partial\dot\phi_\alpha} + \mathrm{CC}$$

in which the A_α, A^*_α are determined from the Lagrange equations (3.12) and the A_i, A^*_i remain arbitrary.

Equation (3.13) defines a constraint submanifold S_1 on which, in the notation of (2.6)

$$X_0 = \dot\phi_i\frac{\partial}{\partial\phi_i} + \dot\phi_\alpha\frac{\partial}{\partial\phi_\alpha} + \mathrm{CC}$$

$$Y_0 \equiv 0.$$

Stability of the SODE conditions under D yield

$$\begin{aligned}D(\eta^{*1}_i) &\equiv d_{ji}A^*_j + d_{\alpha i}A^*_\alpha - r_{ai}\dot\phi^*_a = 0\\ D(\eta^1_i) &\equiv d_{ij}A_j + d_{i\alpha}A_\alpha - r_{ia}\dot\phi_a = 0\end{aligned}\tag{3.14}$$

which determine the A_i, A^*_i, on substituting for A_α, A^*_α from (3.12). The result is

$$\begin{aligned}A^*_i &= r_{aj}d^{-1}_{ji}\dot\phi^*_a - \sum_\alpha d_{\alpha j}d^{-1}_{ji}\left(\frac{1}{m_\alpha}d_{a\alpha}\dot\phi^*_a - \frac{1}{m_\alpha}r_{a\alpha}\phi^*_\alpha\right)\\ A_i &= -r_{ja}d^{-1}_{ij}\dot\phi_a + \sum_\alpha d_{j\alpha}d^{-1}_{ij}\left(\frac{1}{m_\alpha}d_{\alpha\alpha}\dot\phi_a + \frac{1}{m_\alpha}r_{\alpha a}\phi_a\right)\end{aligned}\tag{3.15}$$

and the final vector field solution on S_1 is

$$\begin{aligned}X = {}&\dot\phi_i\frac{\partial}{\partial\phi_i} + \dot\phi^*_i\frac{\partial}{\partial\phi_i} + \dot\phi_\alpha\frac{\partial}{\partial\phi_\alpha} + \dot\phi^*_\alpha\frac{\partial}{\partial\phi_\alpha} + \left(-r_{ja}d^{-1}_{ij}\dot\phi_a + \sum_\alpha d_{j\alpha}d^{-1}_{ij}\left(\frac{d_{\alpha a}}{m_\alpha}\dot\phi_a + \frac{r_{\alpha a}}{m_\alpha}\dot\phi_a\right)\right)\frac{\partial}{\partial\dot\phi_i}\\ &+\left(r_{aj}d^{-1}_{ji}\dot\phi_a - \sum_\alpha d_{\alpha j}d^{-1}_{ji}\left(\frac{d_{a\alpha}}{m_\alpha}\dot\phi^*_a - \frac{r_{a\alpha}}{m_\alpha}\phi^*_a\right)\right)\frac{\partial}{\partial\dot\phi^*_i}\\ &+\sum_\alpha\left(-\frac{d_{\alpha a}}{m_\alpha}\dot\phi_a - \frac{r_{\alpha a}}{m_\alpha}\phi_a\right)\frac{\partial}{\partial\dot\phi_\alpha} + \sum_\alpha\left(\frac{d_{a\alpha}}{m_\alpha}\dot\phi^*_a - \frac{r_{a\alpha}}{m_\alpha}\phi^*_a\right)\frac{\partial}{\partial\dot\phi^*_\alpha}.\end{aligned}\tag{3.16}$$

X yields directly the equation of motion for each of the coordinates ϕ_a, ϕ^*_a. Specifically for the independent degrees of freedom ϕ_α we obtain

$$m_\alpha\ddot\phi_\alpha = (d_{\alpha i}d_{j\beta}d^{-1}_{ji} - d_{\alpha\beta})\dot\phi_\beta + (r_{ja}d^{-1}_{ji}d_{\alpha i} - r_{\alpha a})\phi_a \tag{3.17}$$

on S_1.

Provided the coefficients d and r are well behaved functions of the external field and that $\operatorname{rank}(d_{ij}) = N - r$ for all values of the field then the constraint submanifold S_1 is a continuously deformable submanifold of TQ and the flow of X on S_1 is similarly a well behaved function of the external field—no such problems as acausality arise.

rank$(d_{ij}) = r_1 < N - r$

d_{ij} is anti-Hermitian and may therefore be diagonalized—and without affecting the ϕ_α, ϕ^*_α coordinates. Therefore, introduce matrix S satisfying

$$S^{-1}_{ij} d_{jk} S_{kl} = d_i \delta_{il} \tag{3.18}$$

where

$$d_i = 0 \qquad i = I, J, K, \ldots, = 1, \ldots, (N - r - r_1)$$
$$d_i \neq 0 \qquad i = A, B, C, \ldots, = (N - r - r_1 + 1), \ldots (N - r)$$

and define new quantities

$$\begin{aligned}
&\psi_i = S^{-1}_{ij}\phi_j \qquad \psi^*_i = \phi^*_j S_{ji} \qquad \psi_\alpha = \phi_\alpha \qquad \psi^*_\alpha = \phi^*_\alpha \\
&C_{ij} = S^{-1}_{ik} c_{kl} S_{lj} \qquad \bar{C}_{ij} = S^{-1}_{ik}\bar{c}_{kl} S_{lj} \\
&C_{i\alpha} = S^{-1}_{ij} c_{j\alpha} \qquad C_{\alpha i} = c_{\alpha j} S_{ji} \qquad C_{\alpha\beta} = c_{\alpha\beta} \\
&\bar{C}_{i\alpha} = S^{-1}_{ij}\bar{c}_{j\alpha} \qquad \bar{C}_{\alpha i} = \bar{c}_{\alpha j} S_{ji} \qquad \bar{C}_{\alpha\beta} = \bar{c}_{\alpha\beta} \\
&R_{ij} = S^{-1}_{ik} r_{kl} S_{lj} \qquad R_{i\alpha} = S^{-1}_{ij} r_{j\alpha} \qquad R_{\alpha i} = r_{\alpha j} S_{ji} \qquad R_{\alpha\beta} = r_{\alpha\beta}.
\end{aligned} \tag{3.19}$$

In these variables the Lagrangian (3.4) retains its form

$$\begin{aligned}
L = \sum_\alpha m_\alpha \dot{\psi}^*_\alpha \delta_{\alpha\beta} \dot{\psi}_\beta + \dot{\psi}^*_i C_{ij}\psi_j + \dot{\psi}^*_i C_{i\alpha}\psi_\alpha + \dot{\psi}^*_\alpha C_{\alpha i}\psi_i + \dot{\psi}^*_\alpha C_{\alpha\beta}\psi_\beta - \psi^*_i \bar{C}_{ij}\dot{\psi}_j \\
- \psi^*_i \bar{C}_{i\alpha}\dot{\psi}_\alpha - \psi^*_\alpha \bar{C}_{\alpha i}\dot{\psi}_i - \psi^*_\alpha \bar{C}_{\alpha\beta}\dot{\psi}_\beta - \psi^*_a R_{ab}\psi_b
\end{aligned} \tag{3.20}$$

but is now adapted to the diagonalization of d_{ij}. We will assume that this diagonalization procedure is a well behaved function of the external field.

With

$$D_{ab} = C_{ab} + \bar{C}_{ab} \tag{3.21}$$

the equations determining $\ker\omega$, corresponding to (3.6) are

$$\begin{aligned}
&D_{ji}k^*_j + D_{\alpha i}k^*_\alpha = 0 \\
&-D_{ij}k_j + D_{i\alpha}k_\alpha = 0 \\
&-m_\alpha K^*_\alpha + D_{i\alpha}k^*_i + D_{\beta\alpha}k^*_\beta = 0 \\
&-m_\alpha K_\alpha - D_{\alpha i}k_i - D_{\alpha\beta}k_\beta = 0 \\
&m_\alpha k^*_\alpha = 0 \\
&m_\alpha k_\alpha = 0.
\end{aligned} \tag{3.22}$$

Using the diagonal form of D_{ij} to solve these equations yields, for the most general element of $\ker\omega$:

$$K = \sum_I \left(\frac{\partial}{\partial\psi_I} - \sum_\alpha \frac{D_{\alpha I}}{m_\alpha}\frac{\partial}{\partial\dot{\psi}_\alpha}\right)k_I + \sum_I \left(\frac{\partial}{\partial\psi^*_I} + \sum_\alpha \frac{D_{I\alpha}}{m_\alpha}\frac{\partial}{\partial\dot{\psi}^*_\alpha}\right)k^*_I + K_i\frac{\partial}{\partial\dot{\psi}_i} + K^*_i\frac{\partial}{\partial\dot{\psi}^*_i}$$

where k_I, k_I^*, K_i, K_i^* are all arbitrary. So in this case a basis for $\ker\omega$ is

$$\ker\omega = \left\{ Z_I = \frac{\partial}{\partial\psi_I} - \sum_\alpha \frac{D_{\alpha I}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha},\ Z_I^* = \frac{\partial}{\partial\psi_I^*} + \sum_\alpha \frac{D_{I\alpha}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha^*};\ V_i = \frac{\partial}{\partial\dot\psi_i},\ V_i^* = \frac{\partial}{\partial\dot\psi_i^*} \right\}. \tag{3.23}$$

The general form of $\mathcal{M}$ is unaltered, being independent of d_{ij}, but it is convenient to choose a basis adapted to $\ker\omega$, so we take

$$\mathcal{M} = \left\{ Z_I, Z_I^*, Z_A = \frac{\partial}{\partial\psi_A}, Z_A^* = \frac{\partial}{\partial\psi_A^*};\ W_a = \frac{\partial}{\partial\dot\phi_a},\ W_a^* = \frac{\partial}{\partial\dot\phi_a^*} \right\} \tag{3.24}$$

with Z_I, Z_I^* as for $\ker\omega$.

In the present case the Z_I, Z_I^* give first generation compatibility constraints, absent in the $|d_{ij}| \neq 0$ case. These are $i_{Z_I}\,\mathrm{d}E$, $i_{Z_I^*}\,\mathrm{d}E$ and yield

$$\begin{aligned} \zeta_I^{*1} &= -D_{\alpha I}\dot\psi_\alpha^* + R_{aI}\psi_a^* = 0 \qquad I = 1, 2, \ldots, (N - r - r_1) \\ \zeta_I^1 &= -D_{I\alpha}\dot\psi_\alpha + R_{Ia}\psi_a = 0 \qquad I = 1, 2, \ldots, (N - r - r_1) \end{aligned} \tag{3.25}$$

(3.25) define a submanifold $P_1 \subset TQ$.

The Lagrangian equations for the general vector field

$$X = a_I\frac{\partial}{\partial\psi_I} + a_A\frac{\partial}{\partial\psi_A} + a_\alpha\frac{\partial}{\partial\psi_\alpha} + A_I\frac{\partial}{\partial\dot\psi_I} + A_A\frac{\partial}{\partial\dot\psi_A} + A_\alpha\frac{\partial}{\partial\dot\psi_\alpha} + \mathrm{cc} \tag{3.26}$$

on P_1 give

$$\begin{aligned} a_\alpha &= \dot\psi_\alpha \\ a_\alpha^* &= \dot\psi_\alpha^* \end{aligned} \qquad \forall\alpha \tag{3.27}$$

$$\begin{aligned} -D_{I\alpha}a_\alpha &= R_{Ia}\psi_a \\ D_{\alpha I}a_\alpha^* &= R_{aI}\psi_a^* \end{aligned} \qquad \forall I \tag{3.28}$$

$$\begin{aligned} -d_Aa_A - D_{A\alpha}a_\alpha &= R_{Aa}\psi_a \\ d_Aa_A^* + D_{\alpha A}a_\alpha^* &= R_{aA}\psi_a^* \end{aligned} \qquad \forall A \tag{3.29}$$

$$\begin{aligned} -m_\alpha A_\alpha - D_{\alpha I}a_I - D_{\alpha A}a_A - D_{\alpha\beta}a_\beta &= R_{\alpha a}\psi_a \\ -m_\alpha A_\alpha^* + D_{I\alpha}a_I^* + D_{A\alpha}a_A^* + D_{\beta\alpha}a_\beta^* &= R_{a\alpha}\psi_\alpha^* \end{aligned} \qquad \forall\alpha \tag{3.30}$$

(3.27) determine a_α, a_α^*, (3.28) repeat the compatibility constraints (3.25). Equations (3.29) then determine a_A, a_A^*. The a_I, a_I^* remain undetermined, as do the A_I, A_I^* while the A_α, A_α^* are determined by (3.30) once a_I, a_I^* are known.

The only elements of $\mathcal{M}$ which are not contained in $\ker\omega$ are Z_A, Z_A^* and these yield the SODE conditions

$$\begin{aligned} \eta_A^{*1} &\equiv d_A\dot\psi_A^* + D_{\alpha A}\dot\psi_\alpha^* - R_{aA}\psi_a^* = 0 \\ \eta_A^1 &\equiv d_A\dot\psi_A + D_{A\alpha}\dot\psi_\alpha + R_{Aa}\psi_a = 0 \end{aligned} \tag{3.31}$$

following (2.9). These define a new constraint submanifold, S_1. Comparing these with (3.29) gives

$$\begin{aligned} a_A &= \dot\psi_A \\ a_A^* &= \dot\psi_A^*. \end{aligned} \tag{3.32}$$

The a_I, a_I^* remain arbitrary, but may be chosen to ensure that X of (3.36) is SODE. We write the general X, determined from (3.27)–(3.32) in the form:

$$X = X_0 + Y + V \tag{3.33}$$

where

$$X_0 = \sum_A \frac{1}{d_A}(D_{A\alpha}\dot\psi_\alpha + R_{Aa}\psi_a)\frac{\partial}{\partial\psi_A} + \sum_A \frac{1}{d_A}(-D_{\alpha A}\dot\psi_\alpha^* + R_{aA}\psi_a^*)\frac{\partial}{\partial\psi_A^*}$$
$$+\dot\psi_\alpha\frac{\partial}{\partial\psi_\alpha} + \dot\psi_\alpha^*\frac{\partial}{\partial\psi_\alpha^*} + \sum_\alpha\left[-\frac{1}{m_\alpha}(D_{\alpha A}a_A + D_{\alpha\beta}\dot\psi_\beta + R_{\alpha a}\psi_a)\right]\frac{\partial}{\partial\dot\psi_\alpha}$$
$$+\sum_\alpha\frac{1}{m_\alpha}(D_{A\alpha}a_A^* + D_{\beta\alpha}\dot\psi_\beta^* - R_{a\alpha}\psi_a^*)\frac{\partial}{\partial\dot\psi_\alpha^*} \tag{3.34}$$

is a particular SODE solution of the Lagrange equations, due to (3.31),

$$Y = \sum_I a_I\left(\frac{\partial}{\partial\psi_I} - \sum_\alpha\frac{D_{\alpha I}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha}\right) + \sum_I a_I^*\left(\frac{\partial}{\partial\psi_I^*} + \sum_\alpha\frac{D_{I\alpha}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha^*}\right) \tag{3.35}$$

is an element of $\ker\omega$, and

$$V = A_i\frac{\partial}{\partial\dot\psi_i} + A_i^*\frac{\partial}{\partial\dot\psi_i^*}. \tag{3.36}$$

We make X SODE by choosing the particular $Y = Y_0 \in \ker\omega$

$$Y_0 = \sum_I \dot\psi_I\left(\frac{\partial}{\partial\psi_I} - \sum_\alpha\frac{D_{\alpha I}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha}\right) + \sum_I \dot\psi_I^*\left(\frac{\partial}{\partial\psi_I^*} + \sum_\alpha\frac{D_{I\alpha}}{m_\alpha}\frac{\partial}{\partial\dot\psi_\alpha^*}\right). \tag{3.37}$$

Then

$$D = X_0 + Y_0 + V \tag{3.38}$$

is SODE on S_1 as required.

In D the A_i, A_i^* remain undetermined. This is remedied by the stability conditions. Stability of the non-dynamical SODE conditions (3.31) give the A_A, A_A^* immediately:

$$A_A^* = -\frac{1}{d_A}\sum_\alpha\frac{D_{\alpha A}}{m_\alpha}(D_{I\alpha}\dot\psi_I^* + D_{B\alpha}a_B^* + D_{\beta\alpha}\dot\psi_\beta^* - R_{a\alpha}\psi_a^*) + \frac{R_{aA}}{d_A}\dot\psi_a^* \tag{3.39}$$

from $D(\eta_A^{*1}) = 0$ and

$$A_A = \frac{1}{d_A}\sum_\alpha\frac{D_{A\alpha}}{m_\alpha}(D_{\alpha I}\dot\psi_I + D_{\alpha B}a_B + D_{\alpha\beta}\dot\psi_\beta + R_{\alpha a}\psi_a) - \frac{R_{Aa}}{d_A}\dot\psi_a$$

from $D(\eta_A^1) = 0$, where the a_B, a_B^* are to be substituted from (3.29).

The stability of the compatibility constraints (3.25) yields a second generation of non-dynamical constraints (for example, ζ_I^1 yields $(X_0 + Y_0)(\zeta_I^1) = 0$ with $Y_0(\zeta_I^1) \neq 0$—cf (2.19)):

$$D(\zeta_I^{*1}) = \eta_I^{*2} \equiv O_{aI}\dot\psi_a^* + \sum_\alpha\frac{D_{\alpha I}R_{a\alpha}}{m_\alpha}\psi_a^* = 0$$
$$D(\zeta_I^1) = \eta_I^2 \equiv O_{Ia}\dot\psi_a - \sum_\alpha\frac{D_{I\alpha}R_{\alpha a}}{m_\alpha}\psi_a = 0 \tag{3.40}$$

where

$$O_{\alpha I}=R_{\alpha I}-\sum_{\alpha}\frac{D_{\alpha I}D_{a\alpha}}{m_\alpha}\qquad O_{Ia}=R_{Ia}-\sum_{\alpha}\frac{D_{I\alpha}D_{\alpha a}}{m_\alpha}.$$

These constraints, defining a new submanifold, S_2, contain $\dot\psi_I$, $\dot\psi_I^*$ and so their stability yields equations for A_I, A_I^*:

$$\begin{aligned}&O_{JI}A_J^*+O_{AI}A_A^*+\sum_\alpha\frac{D_{\alpha I}R_{a\alpha}}{m_\alpha}\dot\psi_a^*+\sum_\alpha\frac{O_{\alpha I}}{m_\alpha}(D_{a\alpha}\dot\psi_a^*-R_{a\alpha}\psi_a^*)=0\\&O_{IJ}A_J+O_{IA}A_A-\sum_\alpha\frac{D_{I\alpha}R_{\alpha a}}{m_\alpha}\dot\psi_a-\sum_\alpha\frac{O_{I\alpha}}{m_\alpha}(D_{\alpha a}\dot\psi_a+R_{\alpha a}\psi_a)=0.\end{aligned}\tag{3.41}$$

Up to this point, provided the usual assumptions that $\dim(\ker\omega)=$ constant and the good behaviour of the diagonalization process (3.18), no problems will have been occasioned by the value of the external field—the constraint analysis so far is unaffected by the external field (except that the submanifolds P_1, S_1, S_2 and the flows on them will be continuously deformed under changes in the external field).

At this stage in the analysis however, we see that A_I, A_I^* will only be uniquely determined from (3.41) provided O_{IJ} is non-singular. Even if it is non-singular in the free field theory—i.e. coefficients d, r independent of the external field—it will, in general, still be a function of the external field and may therefore be singular for some values. If O_{IJ} becomes singular then some of the $\dot\psi_I$, $\dot\psi_I^*$ will be undetermined by (3.40) and so some of the A_I, A_I^* will be undetermined and a third generation of constraints is foreshadowed. Since we cannot allow the constraint structure to be dependent on the external field (otherwise for example, the system may lose degrees of freedom for certain values of the external field), we must insist that $\mathrm{rank}(O_{IJ})=$ constant, independent of the external field.

If $|O_{IJ}|=0$ then we must continue the constraint algorithm described in section 2, leading to higher generation of constraints. In principle, nothing new is entailed.

If $|O_{IJ}|\neq 0$ the analysis is now complete, the final vector field solution is uniquely defined on the final constraint submanifold S_2 by the coefficients a_a, a_a^*, A_a, A_a^*, with the A_I, A_I^* given by (3.41). The 'true equations of motion' for the coordinates ψ_a, ψ_a^* are then given by

$$\begin{aligned}&\ddot\psi_a=A_a\\&\ddot\psi_a^*=A_a^*.\end{aligned}\tag{3.42}$$

The independent components on the submanifold S_2 are ψ_α, ψ_α^* and their equations of motion are, for example:

$$\begin{aligned}\ddot\psi_\alpha&=A_\alpha\\&=-\frac{1}{m_\alpha}(a_ID_{\alpha I}+a_AD_{\alpha A}+D_{\alpha\beta}\dot\psi_\beta-R_{\alpha a}\psi_a).\end{aligned}\tag{3.43}$$

Substituting for a_A from (3.29) and $a_I=\dot\psi_I$ from (3.41) yields

$$\begin{aligned}\ddot\psi_\alpha=&\frac{1}{m_\alpha}\left(D_{\alpha I}O_{IJ}^{-1}O_{JA}D_{A\beta}+D_{\alpha I}O_{IJ}^{-1}O_{J\beta}-\frac{D_{\alpha A}D_{A\beta}}{d_A}-D_{\alpha\beta}\right)\dot\psi_\beta\\&+\frac{1}{m_\alpha}\left(\frac{D_{\alpha I}O_{IJ}^{-1}O_{JA}R_{Aa}}{d_A}-\sum_\beta\frac{D_{\alpha I}O_{IJ}^{-1}D_{J\beta}R_{\beta a}}{m_\beta}-\frac{D_{\alpha A}R_{Aa}}{d_A}-R_{\alpha a}\right)\psi_a\end{aligned}\tag{3.44}$$

again provided $|O_{IJ}|\neq 0$.

Equation (3.44) is a somewhat simpler form than the corresponding time independent case of [14] and reflects the relative simplicity of the LR constraint algorithm. We will obtain a similar equation to (3.44) for the conjugate variable ψ^*_α.

4. Conclusions

We have performed the constraint analysis for the general mechanical analogue (3.1) of field theories with constraints, allowing for the presence of time independent external fields in the Lagrangian. The entirely Lagrangian presymplectic geometric algorithm of Lecanda and Román-Roy has been used, to set the constraint analysis and its response to the external field in an intrinsic geometric context. Not only does the algorithm provide a geometric, coordinate independent formulation, but in its geometric view of the compatibility and SODE constraints and their stability, it sets the usually rather *ad hoc* constraint analysis of field theories in a systematic framework, wherein the influence of the external field becomes transparent.

Thus, provided the assumption that dim(ker ω) is constant is upheld for all external field values, *the main determinant of consistency at the first stage of the algorithm* rests on dim(ker ω).

If rank$(d_{ij}) = N - r$ then the bases for ker ω and $\mathcal{M}$ are independent of the external field. There are no compatibility constraints and the stability of the first generation non-dynamical constraints (3.13) which occur guarantees complete determination of the vector field solution of the Lagrange equation. The consistency of this vector field solution is unaffected by the external field.

If rank$(d_{ij}) < N - r$, then we obtain second generation constraints, the analysis of which is facilitated by diagonalizing d_{ij} and we assume that this process is well behaved as a function of the external fields (this is more than a convenience—notwithstanding diagonalization of d_{ij} we would still need to insist that its spectra and eigenspaces behave themselves as functions of the external fields). The null eigenspace of d_{ij} now results in compatibility constraints, (3.25), while SODE conditions (3.31) emerge from the non-singular part. The SODE vector field solutions on the resulting constraint submanifold S_1 (3.38) contains arbitrary vectors V in the vertical sector corresponding to d_{ij}, (3.36). Stability of non-dynamical SODE conditions (3.31) determine those vectors in the non-singular part of d_{ij}, while stability of the compatibility constraints (3.25) yields further non-dynamical constraints (3.40) containing the velocities $\dot{\psi}_I$, $\dot{\psi}^*_I$ corresponding to the null sector of d_{ij}. Thus far the external field has no influence on the consistency of the algorithm. However, at this stage of the second generation non-dynamical constraints obtained from the stability of the first generation compatibility constraints, we meet a constraint typified by (3.40)

$$O_{IJ}\dot{\psi}_I + \ldots = 0$$

with

$$O_{IJ} = R_{IJ} - \sum_\alpha \frac{D_{I\alpha}D_{\alpha J}}{m_\alpha}.$$

Stability of this constraint then yields an equation of the form (3.41)

$$O_{IJ}A_J + O_{IA}A_A - \sum_\alpha \frac{D_{I\alpha}R_{\alpha a}}{m_\alpha}\dot{\psi}_a - \sum_\alpha \frac{O_{\alpha I}}{m_\alpha}(D_{\alpha a}\dot{\psi}_a - R_{\alpha a}\psi_a) = 0.$$

which only determines the final vectors in the solution of the Lagrange equations if $|O_{IJ}| \neq 0$. If $|O_{IJ}| = 0$ we move into a third generation of constraints. At this stage it is imperative for consistency that the external field should not affect the outcome. Thus, if $|O_{IJ}| \neq 0$ in the 'free-field theory' then this must be preserved for all values of the external field. Assuming this is so, (3.41) will allow complete determination of the vector field solution of the Lagrange equations and hence the equations of motion (3.42). The important point is that the singularity of O_{ij} already prejudices the constraint analysis at the level of the second generation constraints and pre-empts any such problems as acausality and quantization in the final 'true equations of motion'. This gives a general confirmation of a similar situation discussed in [16] for the case of the Rarita-Schwinger field.

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